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Existence of the limit at infinity for a function that is integrable on the half line

James Patrick Dix

Abstract. It is well known that for a function that is integrable on $[0, \infty)$, its limit at infinity may not exist. First we illustrated this statement with an example. Then, we present conditions that guarantee the existence of the limit in the following two cases: When the integrable function is non-negative, if the first, second, third, or fourth, derivative is bounded in a neighborhood of each local maximum of f, then the limit exists. Without the non-negative constraint, if an integrable function has a bounded derivative on the entire interval $[0, \infty)$, then the limit exists.

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1 Introduction

As a motivation for this article, we consider the rate of consumption of a non-renewable resource of which only a finite amount is available. Let f be consumption rate. Then the amount consumed of the resource from time zero to time t is represented by $\int_0^t f(s)\,ds$. Since only a finite amount of the resource is available, we have the condition

$$\int_0^\infty f(t)\,dt = A \quad (A \neq \pm\infty)\,, \tag{1.1}$$

which is the main assumption in this work. Does the consumption rate approach a constant (maybe zero) as t approaches infinity? Under what conditions does this limit exist?

Another motivation for our research is considering a continuous probability distribution function defined on the interval $[0,\infty)$, and with finite mean. What is the long time behavior of the probability distribution function?

The discrete version of (1.1) corresponds to a convergent series $\sum_{n=0}^\infty a_n < \infty$; in which case the terms a_n must approach zero as n approaches infinity. However, assumption (1.1) does not guarantee the existence of $\lim_{t\to\infty} f(t)$, as shown in the following example.

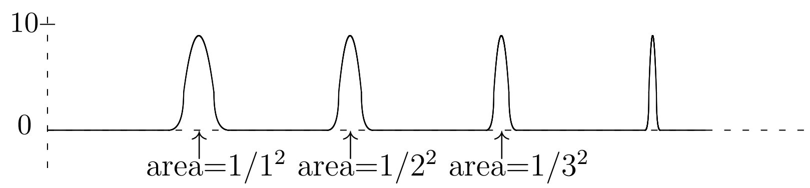

Figure 1: The integral is finite, but $\lim_{t\to\infty} f(t)$ does not exist

Example 1.1. Let $f(t)$ be the function plotted in Figure 1. Note that $\int_0^\infty f(s)\,ds = \sum_{k=1}^\infty 1/k^2 < \infty$ [2, Theorem 9.11]. Therefore, condition (1.1) is satisfied. However, the function $f(t)$ oscillates without approaching any number as t approaches infinity; thus $\lim_{t\to\infty} f(t)$ does not exist. Recall that a function has limit at infinity if and only if its limit superior and its limit inferior are equal [1, Theorem 6.3d]. These two limits are defined as follows:

$$\limsup_{t\to\infty} f(t) = \lim_{t\to\infty} \sup\{u(s) : s \geq t\}, \quad \liminf_{t\to\infty} f(t) = \lim_{t\to\infty} \inf\{u(s) : s \geq t\}\,.$$

In this example, $\limsup_{t\to\infty} f(t) = 10$ and $\liminf_{t\to\infty} f(t) = 0$. Thus $\lim_{t\to\infty} f(t)$ does not exist.

We did not find any references for the existence of this limit; so made our goal to find conditions for the existence of $\lim_{t\to\infty} f(t)$. Our main tools are standard calculus results.

A trivial way to guarantee the existence of $\lim_{t\to\infty} f(t)$ is by assuming that f is monotone. In this case f is of a constant sign on an interval $[t_0,\infty)$. If f is positive, then for satisfying

(1.1), f must be non-increasing and $\lim_{t\to\infty} f(t) = 0$. If f is negative, then for satisfying (1.1), f must be non-decreasing and $\lim_{t\to\infty} f(t) = 0$. If f is identically zero, its limit is zero.

Furthermore, monotonicity of f is implied by a higher-order derivative being of constant sign on an infinite interval. Assume that the nth derivative is of constant sign on an interval $[t_n, \infty)$. Then the $(n-1)$th derivative is monotone and of constant sign on some interval $[t_{n-1}, \infty)$. Then we show that the $(n-2)$th derivative is monotone and of constant sign on some interval $[t_{n-2}, \infty)$. By repeating this process, show that f is monotone on some interval $[t_0, \infty)$.

However, we want to find conditions weaker than the ones above. The existence of the limit of f is established for the following two cases: When f is non-negative, if the first, second, third, or forth derivative is bounded, by the same constant on neighborhoods of all points of relative maximum of f, then $\lim_{t\to\infty} f(t)$ exists; see Theorems 2.3, 2.4, 2.5, and 2.7. When f is allowed to have negative values, if any derivative is bounded on $[0, \infty)$, then $\lim_{t\to\infty} f(t)$ exists; see Theorem 2.9 and Corollary 2.11.

To address probability distribution functions whose mean is finite, we present Corollary 2.12.

2 Results

The following lemmas will be used in proving our main results.

Lemma 2.1. *Let f be continuous on $[0, \infty)$, and $\liminf_{t\to\infty} f(t) \neq \limsup_{t\to\infty} f(t)$. Then there exists a constant α, satisfying $\liminf_{t\to\infty} f(t) < \alpha < \limsup_{t\to\infty} f(t)$, such that for every $\beta > 0$, we can define a sequence of relative maxima $\{f(t_k)\}$ such that*

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \beta \quad \text{for } k = 1, 2, \ldots.$$

Proof. Since the limit superior and the limit inferior are real numbers, not equal to each other, there exists a constant α such that

$$\liminf_{t\to\infty} f(t) < \alpha < \limsup_{t\to\infty} f(t)\,.$$

From the definition of limit superior, there exists a sequence of values of $f(t)$ approaching the limit superior, and another sequence of values of $f(t)$ approaching the limit inferior. Recursively, we define the infinite sequences $\{v_j\}$ and $\{w_j\}$ such that $v_1 < w_1 < v_2 < w_2 < \cdots \to +\infty$, with $f(v_k) > \alpha$ and $f(w_k) < \alpha$.

Since f is continuous on each closed interval $[w_k, w_{k+1}]$, by the extreme value theorem [2, Theorem 3.1], f assumes its maximum at some value $t_k \in (w_k, w_{k+1})$ and

$$f(t_k) \geq f(v_k) > \alpha\,.$$

Note that $\lim_{k\to\infty} t_k = \infty$. Then we define a subsequence, also denoted by $\{t_k\}$, such that

$$t_{k+1} > t_k + \beta \quad \forall k.$$

This completes the proof. □

Lemma 2.2. *Let $f(t)$ be a continuous function for $t \geq 0$, that has a local maximum at value $t_0 > 0$. Then the following items hold:*

(1) If $f'(t_0)$ is defined, then $f'(t_0) = 0$;

(2) If $f''(t_0)$ is defined and f'' is continuous on an interval $[t_k, t_k + \lambda)$ or $(t_k - \lambda, t_k]$, with $\lambda > 0$, then $f''(t_0) \leq 0$;

(3) If $f'''(t_0)$ is defined, then there is no information on the sign of $f'''(t_0)$; it can be positive, negative, or zero.

Proof. Statement (1) follows from [2, Theorem 3.2].

Statement (2) is proven by contradiction. Suppose that $f''(t_0) > 0$. Then by continuity, f'' is positive on an interval containing t_k; therefore, f is concave up on this interval, which contradicts $f(t_k)$ being a local maximum.

Statement (3) is illustrated with examples: $f_1(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$ has a local maximum at $t_0 = 1$ and $f'''(1) = 2$. $f_2(t) = -\frac{1}{3}t^3 + \frac{3}{2}t^2 - 2t$ has a local maximum at $t_0 = 2$ and $f'''(2) = -2$. $f_2(t) = -(t-1)^2$ has a local maximum at $t_0 = 1$ and $f'''(2) = 0$. □

Theorem 2.3. *Assume that $f(t)$ is a continuous and non-negative function satisfying* (1.1). *Also assume that there exist positive constants M and λ such that at each local maximum $f(t_k)$, f is continuous on $[t_k, t_k + \lambda]$ and differentiable on $(t_k, t_k + \lambda)$ satisfying $|f'(t)| \leq M$ on $(t_k, t_k + \lambda)$. Then $\lim_{t\to\infty} f(t)$ exists.*

Proof. By contradiction assume that $\lim_{t\to\infty} f(t)$ does not exist and show the integral in (1.1) approaches $+\infty$.

Since $f \geq 0$, by Lemma 2.1, there exists $\alpha > 0$ and a sequence of local maxima $\{f(t_k)\}$ satisfying

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \beta, \quad \text{with } \beta = \min\{\lambda, \alpha/M\}\,.$$

At each t_k, we draw a right triangle with vertices at $(t_k, 0)$, (t_k, α) and $(t_k, t_k + \beta)$; see Figure 2.

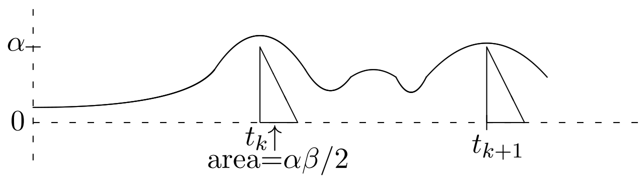

Figure 2: Sequence of local maxima with triangles below

Note that the definition of β does not allow the triangles to overlap, and each triangle has area $\alpha\beta/2$. If the graph of $f(t)$ stays above the hypotenuse of each triangle, then

$$\int_0^\infty f(t)\,dt \geq \sum_{k=1}^\infty \frac{\alpha\beta}{2} = \infty$$

which will complete the proof.

Next we prove that the graph of $f(t)$ stays above the hypotenuse. Note that each hypotenuse has slope with absolute value greater than of equal to M. Since $f(t_k) > \alpha$, if there is a point t, in the base of the triangle, such that $f(t)$ is below or on the hypotenuse, by the Mean Value Theorem there is a point s between t_k and t, where $|f'(s)| > M$. This contradicts the assumption that $|f'(s)| \le M$; therefore $f(t)$ stays above the hypothenuse. This completes the proof. $\square$

Theorem 2.4. *Assume that $f(t)$ is a continuous and non-negative function satisfying* (1.1). *Also assume that there exist positive constants M and λ such that at each local maximum $f(t_k)$, the functions f and f' are continuous on $[t_k, t_k+\lambda]$, and f' is differentiable on $(t_k, t_k+\lambda)$ satisfying $|f''(t)| \le M$ on $(t_k, t_k+\lambda)$. Then $\lim_{t\to\infty} f(t)$ exists.*

Proof. By contradiction assume that $\lim_{t\to\infty} f(t)$ does not exist and show the integral in (1.1) approaches $+\infty$.

Since $f(t) \ge 0$, by Lemma 2.1, there exists $\alpha > 0$ and a sequence of local maxima $\{f(t_k)\}$ satisfying

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \beta, \quad \text{with } \beta = \min\{\lambda, \sqrt{2\alpha/M}\}\,.$$

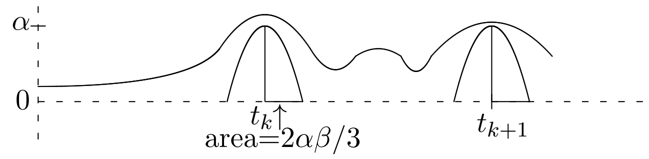

Figure 3: Sequence of local maxima with parabolas

Below each local maximum $(t_k, f(t_k))$, we draw the right branch of the parabola $g(t) = \alpha - \alpha(t-t_k)^2/\beta^2$, with vertex at (t_k, α); see Figure 3. By integrating $g(t)$, from t_k to $t_k+\beta$, we obtain that the area of the "right lobe" of the parabola has value $2\alpha\beta/3$. The definition of β assures that the right lobes of the parabolas do not overlap. If the graph of $f(t)$ stays above the parabola, then

$$\int_0^\infty f(t)\,dt \ge \sum_{k=1}^{\infty} \frac{2\alpha\beta}{3} = \infty,$$

which will complete the proof.

Next we prove that the graph of $f(t)$ stays above the right lobe the parabola. Recall that $f(t_k) > \alpha$, $f'(t_k) = 0$ (because $f(t_k)$ is a local maximum), and $|f''(t)| \le M$. By the Taylor theorem for $f(t)$ at the point t_k with $t_k < t < t_k + \lambda$, there is a point c, $t_k < c < t$, such that

$$\begin{aligned} f(t) &= f(t_k) + f'(t_k)(t-t_k) + f''(c)\frac{(t-t_k)^2}{2} \\ &> \alpha + 0 - M\frac{(t-t_k)^2}{2} \ge \alpha - \frac{\alpha(t-t_k)^2}{\beta^2} = g(t)\,, \end{aligned}$$

which is the parabola defined above. This completes the proof. $\square$

Theorem 2.5. *Assume that $f(t)$ is a continuous and non-negative function satisfying* (1.1). *Also assume that there exist positive constants M and λ such that at each local maximum $f(t_k)$, the functions f, f' and f'' are continuous on $[t_k, t_k + \lambda]$, and f'' is differentiable on $(t_k, t_k + \lambda)$ satisfying $|f'''(t)| \leq M$ on $(t_k, t_k + \lambda)$. Then $\lim_{t\to\infty} f(t)$ exists.*

Proof. By contradiction assume that $\lim_{t\to\infty} f(t)$ does not exist and show that the integral in (1.1) approaches $+\infty$.

Since $f(t) \geq 0$, by Lemma 2.1, there exists $\alpha > 0$ and a sequence of local maxima $\{f(t_k)\}$ satisfying

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \lambda.$$

Note that by Lemma 2.2, $f'(t_k) = 0$ and $f''(t_k) \leq 0$. We consider the only two possible cases for the sequence $\{f''(t_k)\}_{k=1}^{\infty}$.

Case 1: The sequence $\{f''(t_k)\}_{k=1}^{\infty}$ is bounded below by a constant $-\gamma$. Then for $t_k < c < t_k + \lambda$, and using the bound M, we have

$$|f''(c)| = \left|f''(t_k) + \int_{t_k}^{c} f'''(s)\, ds\right| \leq \gamma + M(c - t_k) \leq \gamma + \lambda M\,.$$

By the Taylor theorem for f at t_k with $t_k < t < t_k + \lambda$, there is value c with $t_k < c < t$ such that

$$f(t) = f(t_k) + f'(t_k)(t - t_k) + f''(c)\frac{(t - t_k)^2}{2} > \alpha + 0 - (\gamma + \lambda M)\frac{(t - t_k)^2}{2}\,.$$

When we restrict t to satisfy $|t - t_k| < \lambda$ and $(\gamma + \lambda M)(t - t_k)^2/2 \leq \alpha/2$, we have that $f(t) > \alpha/2$ on infinitely many disjoint intervals of the form $[t_k, t_k + \min\{\lambda, \sqrt{\alpha/(\gamma + M)}\}]$. Then it follows that

$$\int_0^{\infty} f(t) = \infty.$$

This completes the proof for case 1.

Case 2: The sequence $\{f''(t_k)\}_{k=1}^{\infty}$ is unbounded below. Then there are infinitely many k's such that $f''(t_k) < -\lambda M$. Let $[t_k^*, t_k^{**}]$ be the largest interval containing t_k such that $f''(t) \leq 0$ on this interval.

Claim: $t_k^{**} - t_k \geq \lambda$. On the contrary suppose that $t_k^{**} - t_k < \lambda$, then $|f'''(t)| \leq M$ on $[t_k, t_k^{**}]$. Recall that $f''(t_k) < -\lambda M_3$ and $f''(t_k^{**}) = 0$. An application of the mean value theorem yields a value $c \in (t_k, t_k^{**})$ such that $|f'''(c)| > M$. This contradiction proves the claim.

From this claim, the graph of f is concave down on the interval $[t_k, t_k^{**}]$. Then the line that connects $(t_k, f(t_k))$ and $(t_k + \lambda, f(t_k + \lambda))$ lies under the graph of f, so the area under the graph of f and under the straight line satisfy

$$\int_{t_k}^{t_k+\lambda} f(t)\, dt \geq \frac{f(t_k) + f(t_k + \lambda)}{2}\lambda > \frac{0 + \alpha}{2}\lambda\,.$$

The second inequality holds because $f(t_k + \lambda) \geq 0$ and $f(t_k) > \alpha$.

Since $t_k + \lambda < t_{k+1}$, there are infinitely many disjoint intervals, of length λ, where the above estimate holds. It follows that $\int_0^\infty f(t) = \infty$. This proves case 2 and completes the proof. $\square$

Remark 2.6. The assumptions for Theorems 2.3–2.5 are stated on "right" intervals $[t_k, t_k + \lambda]$. However, we can reach the same conclusion using "left" intervals $[t_k - \lambda, t_k]$, or a combination of these two types of intervals.

In the next theorem, we do not have this choice because the third derivative is required to be bounded on both sides of t_k.

Theorem 2.7. *Assume that $f(t)$ is a continuous and non-negative function satisfying* (1.1). *Also assume that there exist positive constants M and λ such that at each local maximum $f(t_k)$, the functions f, f', f'' and f''' are continuous on $[t_k-\lambda, t_k+\lambda]$, and f''' is differentiable on $(t_k - \lambda, t_k + \lambda)$ satisfying $|f^{(4)}(t)| \leq M$ on $(t_k - \lambda, t_k + \lambda)$. Then $\lim_{t\to\infty} f(t)$ exists.*

Proof. By contradiction assume that $\lim_{t\to\infty} f(t)$ does not exist and show that the integral in (1.1) approaches $+\infty$.

By Lemma 2.1, there exists $\alpha > 0$ and a sequence of local maxima $\{f(t_k)\}$ satisfying

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \lambda.$$

By Lemma 2.2, for all k: $f'(t_k) = 0$ and $f''(t_k) \leq 0$. Let $[t_k^*, t_k^{**}]$ be the largest interval containing t_k such that $f''(t) \leq 0$ on this interval. We consider the two possible cases for the lengths of these intervals.

Case 1: $\lim_{k\to\infty} t_k^{**} - t_k^* = 0$. From the definition of limit, there exists an index k_0 such that $0 \leq t_k^{**} - t_k^* < \lambda$ for all $k \geq k_0$; thus $[t_k^*, t_k^{**}] \subset [t_k - \lambda, t_k + \lambda]$.

Claim: $\lim_{k\to\infty} f''(t_k) = 0$. Note that $f''(t_k^*) = f''(t_k^{**}) = 0$ and f'' is continuous on $[t_k^*, t_k^{**}]$. By the extreme value theorem, f'' attains its minimum at some point $c_k \in (t_k^*, t_k^{**})$. Then $f''(c_k) \leq f''(t_k)$. Using that $f'''(c_k) = 0$ and the bound M, for $t \in [t_k - \lambda, t_k + \lambda]$, we have

$$f'''(t) = f'''(c_k) + \int_{c_k}^{t} f^{(4)}(s)\, ds \leq M|t - c_k| \leq 2\lambda M\,.$$

Thus the graph of f'' intersects the t-axis at t_k^* and at t_k^{**}, and has slope bounded by $2\lambda M$. By the fundamental theorem of calculus,

$$|f''(t_k^{**}) - f''(c_k)| = \Big|\int_{c_k}^{t_k^{**}} f'''(t)\, dt\Big| \leq \int_{c_k}^{t_k^{**}} |f'''(t)|\, dt \leq 2\lambda M(t_k^{**} - c_k),$$

and similarly,

$$|f''(c_k) - f''(t_k^*)| \leq 2\lambda M(c_k - t_k^*)\,.$$

Therefore,

$$-2\lambda M(t_k^{**} - c_k) \leq f''(t_k^{**}) - f''(c_k) \leq 2\lambda M(t_k^{**} - c_k),$$
$$-2\lambda M(c_k - t_k^*) \leq f''(c_k) - f''(t_k^*) \leq 2\lambda M(c_k - t_k^*).$$

Since $f''(t_k^{**}) = f''(t_k^*) = 0$, it follows that $|f''(c_k)| \leq 2\lambda M(t_k^{**} - t_k^*)$. Then the inequality $f''(c_k) \leq f''(t_k) \leq 0$ implies $\lim_{k\to\infty} f''(t_k) = 0$, which proves the claim.

From this claim, for each $\beta > 0$, there exists an index k_0 such that $|f''(t_k)| \leq \beta$ for all $k \geq k_0$. By the Taylor theorem, for f at t_k with $t_k - \lambda < t < t_k + \lambda$, we have

$$\begin{aligned} f(t) &= f(t_k) + f'(t_k)(t - t_k) + f''(t_k)\frac{(t - t_k)^2}{2} + f'''(c)\frac{(t - t_k)^3}{6} \\ &> \alpha + 0 - \beta\frac{|t - t_k|^2}{2} - 2\lambda M\frac{|t - t_k|^3}{6} . \end{aligned}$$

By restricting t to satisfy $|t - t_k| < \lambda$, $\beta|t - t_k|^2/2 \leq \alpha/3$ and $2\lambda M|t - t_k|^3/6 \leq \alpha/3$, we have that $f(t) > \alpha/3$ on infinitely many intervals, each one of length $\min\{\lambda, \sqrt{2\alpha/(3\beta)}, \sqrt[3]{3\alpha/(2\lambda M)}\}$. Since these intervals are disjoint, it follows that $\int_0^\infty f(t) = \infty$. This completes the proof for case 1.

Case 2: There exists a positive constant γ such that $t_k^{**} - t_k^* \geq \gamma$ for infinitely many k's. Then the graph of f is concave down on $[t_k^*, t_k^{**}]$. Since $f(t) \geq 0$ and $f(t_k) > \alpha$, the area under the graph can be estimated as

$$\begin{aligned} &\int_{t_k^*}^{t_k} f(t)\, dt + \int_{t_k}^{t_k^{**}} f(t)\, dt \\ &\geq \frac{f(t_k^*) + f(t_k)}{2}(t_k - t_k^*) + \frac{f(t_k^{**}) + f(t_k)}{2}(t_k^{**} - t_k) \\ &> \frac{0 + \alpha}{2}(t_k^{**} - t_k^*) \geq \frac{\alpha\gamma}{2} . \end{aligned}$$

Since intervals $[t_k^*, t_k^{**}]$ cover a set of intervals whose lengths add up to infinity, $\int_0^\infty f(t) = \infty$. This proves case 2. It also completes the proof. $\square$

Remark 2.8. The above methods for proving Theorems 2.3–2.7 does not seem to work when only the fifth derivative is bounded. However, we are unable to find an example where the fifth derivative is bounded, $f(t) \geq 0$, and $\lim_{t\to\infty} f(t)$ does not exist.

Next we consider a function which may have positive and negative values, however we need to assume differentiability on the entire interval $[0, \infty)$.

Theorem 2.9. *Assume that $f(t)$ is a differentiable function satisfying* (1.1)*, and that there exists a positive constant M such that $|f'(t)| \leq M$ on for all $t \geq 0$. Then* $\lim_{t\to\infty} f(t) = 0$.

Proof. To prove that $\lim_{t\to\infty} f(t) = 0$, we show that $\limsup_{t\to\infty} |f(t)| = 0$. By contradiction, assume that $\limsup_{t\to\infty} |f(t)| = \alpha$ for some positive constant α. Then there exists a sequence $f(t_k)$ approaching either α or $-\alpha$, and $\{t_k\} \to \infty$ as $k \to \infty$. First assume that $\lim_{k\to\infty} f(t_k) = \alpha$. As in Lemma 2.1, we select a subsequence such that $t_{k+1} \geq t_k + (2\alpha/(3M))$ and

$$f(t_k) \geq 2\alpha/3, \quad k = 1, 2, \ldots .$$

The mean value theorem and the assumption $|f'(s)| \leq M$ guarantee the existence of an interval I_1 with center t_1 and radius $\alpha/(3M)$ where $f(t) \geq \alpha/3$. Similarly, at t_2, we obtain an interval I_2 with center t_2 and radius $\alpha/(3M)$ where $f(t) \geq \alpha/3$. The same process yields a sequence of non-overlapping intervals $I_1, I_2, I_3, \dots$, each one of length $2\alpha/(3M)$. Let $I_k = [a_k, b_k]$; then a_k and b_k approach infinity as k approaches infinity. Since $f(t) \geq \alpha/3$ on $[a_k, b_k]$, we have

$$\Big|\int_{a_k}^{b_k} f(t)\,dt\Big| = \int_{a_k}^{b_k} f(t)\,dt \geq \frac{2\alpha^2}{9M}\,.$$

However, from (1.1), for $2\alpha^2/(9M) > 0$ there exists a value T such that $a_k, b_k \geq T$ implies

$$\Big|\int_0^{b_k} f(t)\,dt - \int_0^{a_k} f(t)\,dt\Big| = \Big|\int_{a_k}^{b_k} f(t)\,dt\Big| < \frac{2\alpha^2}{9M}\,.$$

This contradiction implies $\alpha = 0$.

Now assume that $\lim_{k\to\infty} f(t_k) = -\alpha$ Note that $\lim_{k\to\infty}(-f(t_k)) = \alpha$, and that the function $-f$ satisfies (1.1) with $-A$ instead of A. Using the first part of this proof, for $-f$, we conclude that $\alpha = 0$. This completes the proof. □

Theorem 2.10. *Assume that $f(t)$ satisfies (1.1), and has continuous derivatives on $[0,\infty)$. If the k-th derivative satisfies $\sup_{t\geq 0}|f^{(k)}(t)| = \infty$, then $\sup_{t\geq 0}|f^{(p)}(t)| = \infty$ for all $p \geq k$.*

Proof. On the contrary suppose that for some integer n, $\sup_{t\geq 0}|f^{(n)}(t)| = \infty$, and that for all $t \geq 0$ there exists a constant M such that $|f^{(n+1)}(t)| \leq M$. Then for the value

$$\alpha = M\big(2^n + 2^{n-1} + \cdots + 2^1 + 2^0 + 1\big)$$

there exists a t_1 such that $|f^{(n)}(t_1)| \geq \alpha + 1$. Using that the absolute value of the slope of $f^{(n)}$ is bounded by M, by the mean value theorem, there exists an interval I_n, with center t_1 with radius α/M such that $|f^{(n)}(t)| \geq 1$ for all $t \in I_n$.

On I_n, the function $f^{(n-1)}$ is strictly monotonic because the absolute value of its derivative is greater than or equal to 1. Let

$$\widetilde{I}_n = \{t \in I_n : |f^{(n-1)}(t)| < 1\}\,.$$

This set can be empty, or an interval of length at most 2, because the absolute value of the slope of $f^{(n-1)}$ is at least 1.

If the set $I_n \setminus \widetilde{I}_n$ consists of a single subinterval of I_n, we let $I_{n-1} = I_n \setminus \widetilde{I}_n$. (In set notation $A \setminus B$ consists of all the elements that are in A but not in B).

If the set $I_n \setminus \widetilde{I}_n$ consists of two subintervals, we let I_{n-1} be larger of the two subintervals. In both cases $I_{n-1} \subset I_n$, and I_{n-1} is an interval of length at least $\frac{1}{2}(\frac{2\alpha}{M} - 2)$.

On I_{n-1}, the function $f^{(n-2)}$ is strictly monotonic and the absolute value of its slope is greater than or equal to 1. Let $\widetilde{I}_{n-1} = \{t \in I_{n-1} : |f^{(n-2)}(t)| < 1\}$. Repeating the above process, we define an interval $I_{n-2} \subset I_{n-1}$ whose length is at least $\frac{1}{2}\big(\frac{1}{2}(\frac{2\alpha}{M} - 2) - 2\big)$.

Repeating this process, n times, we obtain a interval I_0 whose length is at least 2, such that $|f(t)| \geq 1$ for all $t \in I_0$. Let $[a_1, b_1] = I_0$.

Since $f^{(n)}$ is continuous and unbounded on $[0, \infty)$, for the same α defined above, there exists a $t_2 \geq t_1 + (2\alpha/M)$ such that $|f^{(n)}(t_2)| \geq \alpha + 1$. As above we obtain an interval $[a_2, b_2]$ of length at least 2, such that $|f(t)| \geq 1$ for all $t \in [a_2, b_2]$.

Repeating the above process we obtain a sequence of non-overlapping intervals $[a_k, b_k]$, each one of length at least 2. Since $|f(t)| \geq 1$ and is continuous on $[a_k, b_k]$, we have

$$\Big| \int_{a_k}^{b_k} f(t)\, dt \Big| = \int_{a_k}^{b_k} |f(t)|\, dt \geq 2\,.$$

However, from (1.1), for the positive number 2 there exists a value T such that $a_k, b_k \geq T$ implies

$$\Big| \int_0^{b_k} f(t)\, dt - \int_0^{a_k} f(t)\, dt \Big| = \Big| \int_{a_k}^{b_k} f(t)\, dt \Big| < 2\,.$$

This contradiction implies that $f^{(n+1)}$ can not be bounded. The proof is complete. $\square$

As a consequence of Theorems 2.9 and 2.10, we have the following result for functions that may have negative values.

Corollary 2.11. *Assume that $f(t)$ satisfies (1.1), and has continuous derivatives. If the k-th derivative satisfies $\sup_{t\geq 0} |f^{(n)}(t)| < \infty$, then $\sup_{t\geq 0} |f^{(n)}(t)| < \infty$ for all $k \leq n$. Furthermore, $\lim_{t\to\infty} f(t) = 0$.*

We conclude this section by considering a continuous probability distribution function defined on the interval $[0, \infty)$, and having finite mean.

Corollary 2.12. *Assume that the probability distribution function g is defined on the interval $[0, \infty)$ and its mean satisfies*

$$\int_0^\infty t g(t)\, dt = \mu < \infty\,.$$

If for any pair of derivatives $g^{(n)}$, $g^{(n+1)}$, we have $\sup_{t\geq 0} |g^{(n)}(t) + t g^{(n+1)}(t)| < \infty$, then

$$\lim_{t\to\infty} t g(t) = 0.$$

Proof. By setting $f(t) = t g(t)$ in Corollary 2.11, it follows that if for some n, $\sup_{t\geq 0} |g^{(n)}(t) + t g^{(n+1)}(t)| < \infty$. Then $\lim_{t\to\infty} t g(t) = 0$. $\square$

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